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J. Phys. A: Math. Gen. 27 (1994) L237–L242. Printed in the UK

LETTER TO THE EDITOR

Antipodal correlations and the texture (fractal lacunarity) in critical percolation clusters

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Received 21 February 1994

Abstract. The antipodal correlations between opposing sectors of large critical percolation clusters are negative, as expected, but vanish in the infinite size limit approximated by an inner square much smaller than the whole lattice but much larger than the nearest-neighbour distance. Antipodal correlations are a new numerical measure of fractal lacunarity, that is, of texture: zero correlation expressed that the lacunarity is neutral in percolation clusters.

Of all the random fractals [1] of interest to physics, the most thoroughly understood are the critical percolation clusters [2]. Therefore, we chose those clusters to test recent advances in the numerical study of fractals' texture, more specifically of their 'lacunarity', a notion concerned with departure from translational invariance, and the size distribution of the holes [3].

As mentioned in [1] (p 313), the shape of a Cantor set is far from being fully determined by its fractal dimension D. Thus, figure 1 illustrates a stack of Cantor sets; all are of dimension $\log N/\log(1/r) = \log 2^k/\log 4^k = 1/2$, but k varies from 1 in the middle line to 10 as one moves up or down. The 2^k intervals of this generator are either distributed uniformly (below the middle line) or split between two tight groups (above the middle line). Only the middle lines 'look fractal', the bottom one 'looks like' a filled interval and the top one seems to reduce to the endpoints of an interval.

On every line, every level of approximation is made of small intervals of identical lengths; therefore the interpolation used in figure 1 can be immediately replaced by a completely equivalent extrapolation process.

To appreciate the range of texture compatible $D = 1.89$, compare the central portions of a large critical cluster with plates 318 left and 318 right of [1]. The cluster is less 'coarse', closer to translational invariance than plate 318 right and, *a forteriori*, plate 318 left.

Lacunarity is *not* a number but a complex notion that demands several numbers (not functions of each other) to be fully grasped [3]. Cantor sets, and all fractals with a very strict hierarchy, are hardest to study analytically, as [3] shows for one measure.

However, our study of the lacunarity of percolation clusters has so far gone in a different direction. The underlying fact was discovered recently by one of us [3]: the 'antipodal correlation' $C(\pi)$—*to be defined—affects lacunarity, therefore can be added to the existing* measures of lacunarity. This discovery relates to the intuition that, overall, plane fractals with a dimension close to 2 'tend' to have small holes, and plane fractals with a dimension

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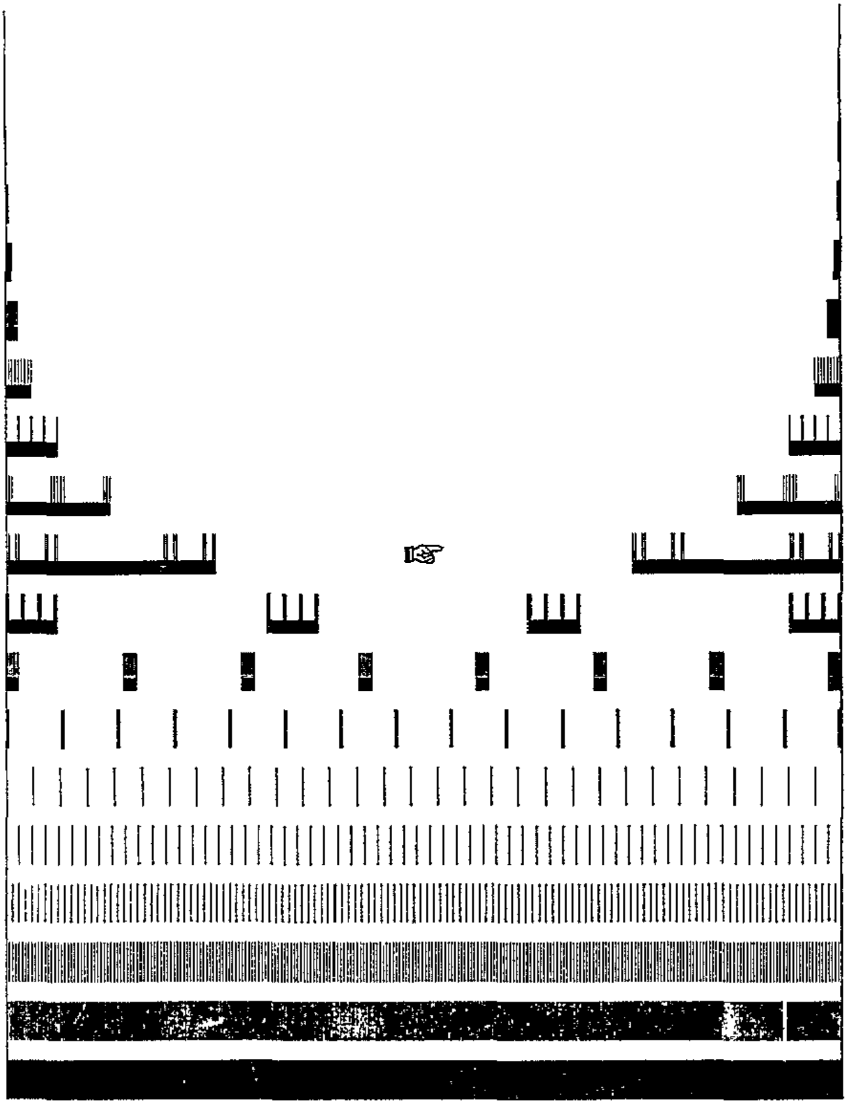

Figure 1. A stack of Cantor sets of common dimension $D = 1/2$. The lacunarity is low near the bottom, high near the top and 'neutral' near the middle line of the stack. (By construction, all these sets are self-similar, and have the same fractal dimension $D = 1/2$. Yet, as one moves from bottom to top, one seems to move from a full unit interval that has been hollowed out, so that only the end points are left in.)

close to 0 'tend' to have enormous holes. It is now possible, for each D, to define fractals of *neutral lacunarity* by the condition that $C(\pi) = 0$, and to measure low or high lacunarity by the value of $C(\pi)$ (with a few added complications). Antipodal correlation will now be defined, explained, and tested, first, on some fully controllable fractals called trema sets [1] (chapters 33 and 34) and, second, on critical percolation clusters.

To define the correlation, start with a very large critical cluster, and consider the sites in a small box $R \times R$ whose centre is a site in the clusters. Consider two 'pie slice' sectors, each bounded by two straight half-lines from the centre, making an angle of ϕ, and denote by θ the angle between the pie slices' centre lines. To obtain the correlation between the numbers of sites in two sectors, we count the occupied sites in those sectors, call them x and y, and evaluate

$$C(\theta) = (\langle xy\rangle - \langle x\rangle\langle y\rangle)/(\langle x^2\rangle - \langle x\rangle\langle x\rangle)$$

where $\langle\cdots\rangle$ denotes an average over many realizations of a random fractal. This correlation will depend on θ and ϕ. *Antipodal* correlations are those for opposite sectors, $\theta = \pi$.

The empirical finding in this letter (on which we elaborate momentarily) is that, under suitable limit conditions, the numbers x and y are uncorrelated. This is shown in figures 2 and 3.

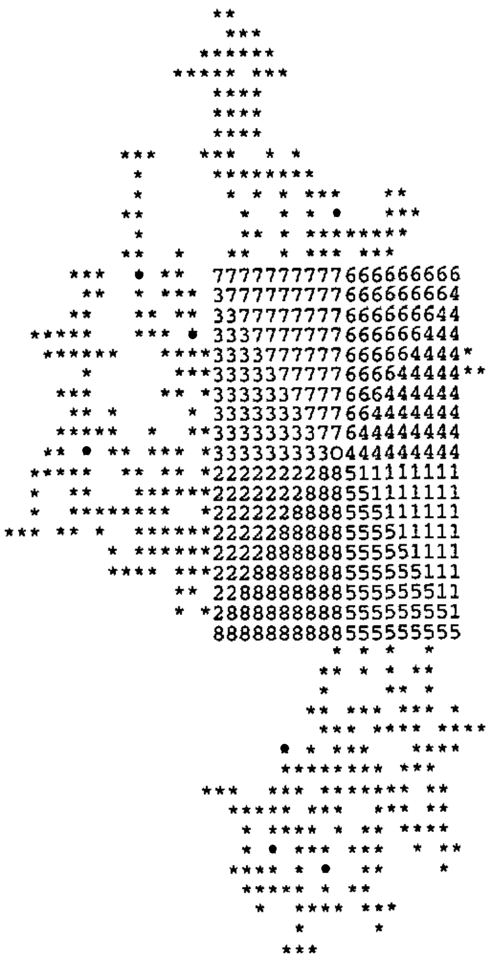

Figure 2. Illustration of a critical cluster, centre square, and sectors to be analysed; $L = 51$, $R = 10$. Boundary sites between adjacent sectors count for both sectors but in the figure only the larger of the two numbers is shown.

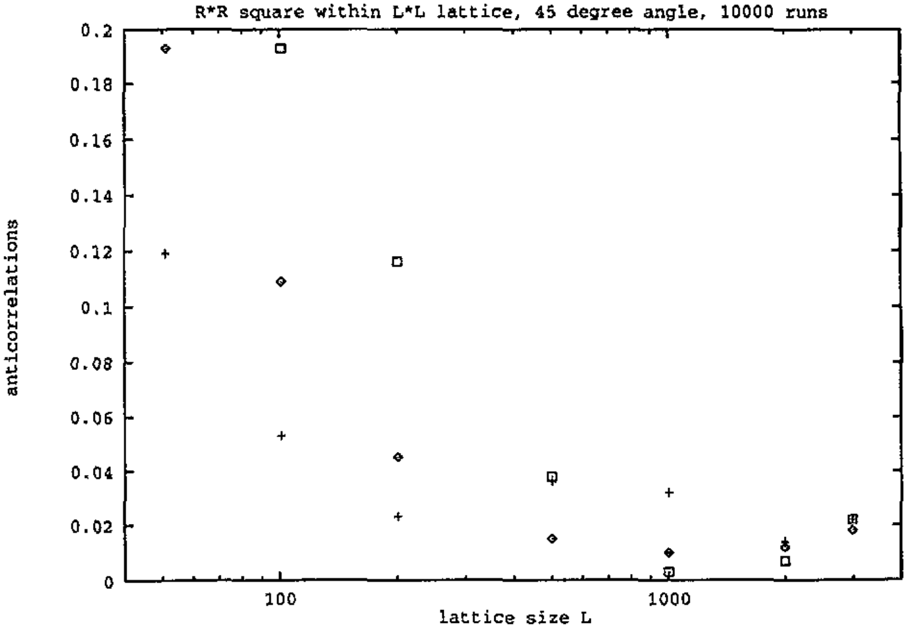

Figure 3. Absolute value of the anticorrelation $-C(\theta)$ for the antipodes ($\theta = \pi$) as a function of lattice size, for a centre of size $R = 19$ (+), $R = 39$ (◊) and $R = 79$ (□).

Since the question has not been previously raised, one might *a priori* wonder whether or not this result was to be expected. The fact that it was not contrary to expectations is difficult to establish in the case of figure 1, because of the strict hierarchy built into Cantor sets. An intuitive feeling for the meaning of antipodal independence is easiest to obtain using the objects with holes shown on plates 306 to 309 of [1]. They are examples of 'trema sets' produced by cutting out from the plane circular 'tremas' (=holes) with a suitable scaling distribution of radii and random positions of the centres of the circles. Different circles are allowed to overlap, and the remainder *not* covered by any of the circles defines the trema set, which looks like Swiss cheese.

In this structure, the antipodal correlation vanishes in the limit of small cone angles ϕ. Indeed, a circular hole cut out from the plane cannot overlap with a sector if it overlaps already with the opposite sector. It follows that the antipodal correlations are zero for the trema sets of $\phi \to 0$. (Had we not required the origin to be occupied, a hole could cover the origin and cut through all sectors, thus destroying this property of non-correlation.) Correlations are possible only for larger cone angles ϕ or for sectors that are not exactly opposite $\theta < \pi$.

If the tremas are non-convex, the value of D is unchanged, but it is possible for antipodal lines to be intersected by the same trema. This creates *positive* antipodal correlations, particularly when the tremas are thin circular annuli. (Annular tremas were selected to model rain clouds, rather successfully [4], well before the significance of antipodal correlations was recognized explicitly). Thus, making the trema non-convex does not change D but changes the 'swiss cheese' to have smaller holes, hence a smoother, 'less fractal' texture.

Needle-shaped tremas (see plates 323 in [1]) are convex, hence yield no antipodal correlation and $C(\pi) = 0$, but the value of $C(\theta)$ for θ slightly different from π is much higher than for circular tremas, and the result is much lower lacunarity.

Different manipulations of the trema cause the holes to become bigger and create a negative correlation.

The preceding example of the trema sets suffices to establish that antipodal non-correlation is not an obvious property, but one that characterizes numerically certain very special fractals. The fact that $C(\pi) \neq 0$ is possible means that the long range dependence that is characteristic of a fractal is compatible with diverse levels of detailed dependence. Absence of correlation for θ near π expressed that the origin's being occupied creates a kind of 'screening' or 'shielding' between near-antipodal directions, and expressed that a fractal is as relaxed or unconstrained and local as can be.

Though percolation is a completely random procedure, all parts of the percolating cluster have to be connected, creating correlations not present in the trema sets. Now we describe our Monte Carlo simulations for site percolation. The probability of each site being empty or occupied is the same for every site and independent of the other sites. At the percolation threshold, an infinite cluster of neighbouring occupied sites is formed for the first time [2]. It is often called the critical, or the incipient infinite, cluster. This critical cluster is a fractal with odd-shaped holes. We looked at critical percolation clusters on 10 000 $L \times L$ square lattices, with L varying from 13 to 3001. Clusters were grown with the Leath growth algorithm starting from the occupied origin of the lattice [5]. Only clusters reaching one of the four boundaries were analysed. We did not investigate how the different weighting of other algorithms (like Hoshen–Kopelman) would affect our results; we found that our numbers change if we demand of the cluster that it touches the upper or lower boundary.

To avoid the lattice anisotropy (which we think will yield different results for different angles), we always worked with the eight sectors having one of the lattice axes as a boundary. Sites on a boundary between two sectors were counted fully for both sectors; for $\phi < \pi/4$, the sites not belonging to a sector were not analysed. Figure 2 gives a percolation example with a division into eight sectors of 45 degrees each. Angles smaller than $\phi = \pi/4$ were realized by taking $\phi = \tan^{-1}(1/n)$, with $n = 1, 2, 3, \ldots, 10$. For example, the first sector for positive x coordinates and positive small y coordinates has x varying from 0 to $L/2$, and for a given x, the y coordinate varies from 0 to x/n and is an integer rounded downward.

By definition, $C(\theta) = 1$ when $\theta = 0$. For adjacent sectors we found $C(\theta)$ to be positive and to approach unity when n increases (narrow cones). Antipodal sectors, however, showed negative correlations; this means that in this method a cluster extending strongly to the right has less than usual mass on its left. (This effect is reduced somewhat if we demand that, besides touching one of the four outer boundaries, the cluster also touches all found boundaries of an inner square of size $(L/3)^2$ centred about the origin.) However, the anticorrelations between the antipodes become very small if we do not analyse the whole cluster but only centred $R\times$ squares. Only here can be expected simple power laws and self-similarity to apply, just as for diffusion limited aggregates and random walks [6]. Thus within the large $L \times L$ lattice which the cluster has to span, we counted only the cluster sites in a small square of size $R \times R$. Now for fixed R and $L \to \infty$, figure 2 shows that the antipodal anticorrelation C decays towards a value close to the statistical error ~ 0.01; we regard the minima at and above $L = 1000$ as random fluctuations. (However, the antipodal correlations are negative for all L; hence, the statistical significance of the plateau seen in figure 3 for large L deserves a closer look.)

The comparison of our data in figure 3 for $R = 19$, 39 and 79 shows that the larger R yield a more pronounced decay of the anticorrelation towards a smaller value. Plotted versus L/R instead of L, the three data sets roughly collapse in the decay region (but not in the final plateau of uncertain statistical significance). Thus in the scaling limit

$$a \ll R \ll L$$

(where a is the distance between nearest neighbours) the correlations between antipodes vanish.

The main result, that the antipodes are anticorrelated for $L = R$ and become decorrelated for $L \gg R$, does not depend qualitatively on the lattice size or the cone angle $\phi = \tan^{-1}(1/n)$. Decreasing this cone angle from 45 degrees, the antipodal correlations for the hole cluster ($R = L$) decrease slightly from -0.22 at $n = 1$ to -0.19 at $n = 10$ (and to 0.17_5 for infinite n, which restricts attention to the sites that are on the lattice axes). Also, correlations between nearly opposite sectors, $\theta = \pi \pm \phi$, approach those for opposite sectors if the angle ϕ becomes small. And for $R = l$ and $\phi = \pi/4$, the antipodal correlations varied between -0.213 and -0.225 for $L = 51$, 101, 251, 501, 1251, 1601 and 2001.

In conclusion, we showed that antipodes are anticorrelated for large critical percolation clusters on the square lattice, but anticorrelation vanishes in the central part of the clusters. Slices need not be very thin for the anticorrelation to vanish.

One of us (DS) thanks the Canada Council and the German–Israeli foundation for support and Naeem Jan for discussions. We thank Amnon Ahorony for very useful comments and the use of his workstation.

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